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PARTITION RELATIONS FOR UNCOUNTABLE ORDINALS

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ABSTRACT

Partition relations of the form $\alpha \rightarrow (\alpha, m)^2$, where α is an ordinal and m is a positive integer, are considered. Let κ be a cardinal. The following are proved: If κ is singular and $2^\kappa = \kappa^+$ then $(\kappa^+)^2 \nrightarrow ((\kappa^+)^2, 3)^2$. If κ is a strong limit cardinal, then $\kappa^2 \rightarrow (\kappa^2, m)^2$ iff $((\mathrm{cf}\kappa)^2 \rightarrow ((\mathrm{cf}\kappa)^2, m)^2$. If κ is regular and $\kappa^2 \rightarrow (\kappa^2, 3)^2$, then the κ-Souslin hypothesis holds. If $\kappa^\omega < \alpha < \kappa^+$ and $\mathrm{cf}\alpha = \mathrm{cf}\kappa > \omega$, then $\alpha \nrightarrow (\alpha, 3)^2$.

1. Preliminaries

Our set-theoretic usage is fairly standard. Each ordinal is identified with the set of its predecessors. Since the axiom of choice is assumed throughout, cardinals are identified with initial ordinals. We use ω_α to denote the initial ordinal occupying position α in the sequence of all initial ordinals; of course $\omega_0 = \omega$. If κ is a cardinal then κ^+ is the next largest cardinal. The abbreviations ZFC, GCH and $V = L$ stand for Zermelo-Frankel set theory with the axiom of choice, the generalized continuum hypothesis, and Gödel's axiom of constructibility, respectively.

All exponentiation in this paper is to be interpreted as *ordinal* exponentiation.

If X is a set then $|X|$ is the cardinality of X and $[X]^n$ is the set of all n-element subsets of X. If f is a function whose domain includes X, then $f \mid X$ denotes the restriction of f to X.

If α is an ordinal then $\mathrm{cf}\alpha$ denotes the least ordinal β which can be mapped onto a cofinal subset of α. The ordinal α is said to be *indecomposable* iff α cannot be represented as a sum $\beta + \gamma$ where $\beta, \gamma < \alpha$. It is well-known (see

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[14]) that if α is indecomposable and $\alpha = B \cup C$, then either B or C has order type α. The indecomposable ordinals are precisely the ordinals of the form ω^β for some ordinal β.

Let κ be a regular uncountable cardinal. A set $C \subseteq \kappa$ is said to be *closed unbounded* (in κ) provided that C is cofinal in κ and for every $\alpha < \kappa$, $\sup(C \cap \alpha) \in C$. It is well known that if C_α is closed unbounded for every $\alpha < \kappa$, then so is $\{\beta < \kappa\colon \text{for all } \alpha < \beta,\ \beta \in C_\alpha\}$.

A partial ordering $(T, \leqq_T)$ is a *tree* provided that for all $s \in T$, $\{t \in T\colon t <_T s\}$ is well-ordered by $\leqq_T$. The *level* of s, written $l(s)$, is the order type of $\{t \in T\colon t <_T s\}$. Let κ be a regular cardinal. A tree $(T, \leqq_T)$ is a *κ-Souslin tree* if it satisfies the following conditions:

(1) $|T| = \kappa$,

(2) if $s \in T$ and $l(s) = \alpha$ then there are $t_1, t_2 \in T$ such that $t_1 \neq t_2$, $l(t_1) = l(t_2) = \alpha + 1$ and $s <_T t_1, t_2$

(3) if $s \in T$ then $|\{t \in T\colon s <_T t\}| = \kappa$

(4) if $A \subseteq T$ is a set of pairwise incomparable elements, then $|A| < \kappa$.

A set $B \subseteq T$ is called a *branch* if B is totally ordered by $<_T$. Using conditions (2) and (4), it is easy to see that a κ-Souslin tree can have no branches of power κ. The *κ-Souslin hypothesis* is the assertion that there are no κ-Souslin trees.

Suppose $\alpha, \beta_0, \beta_1, \cdots, \beta_{m-1}$ are ordinals and $n < \omega$. The partition symbol

$$\alpha \to (\beta_0, \beta_1, \cdots, \beta_{m-1})^n$$

means that for any $f\colon [\alpha]^n \to \{0, 1, \cdots, m-1\}$ there exist $i < m$ and $X \subseteq \alpha$ such that X has order type β_i and $f(x) = i$ for all $x \in [X]^n$. Alternately, $\alpha \to (\beta_0, \cdots, \beta_{m-1})^n$ holds iff whenever $[\alpha]^n = P_0 \cup \cdots \cup P_{m-1}$ then there are $i < m$ and $X \subseteq \alpha$ such that X has type β_i and $[X]^n \subseteq P_i$. The negation of $\alpha \to (\beta_0, \cdots, \beta_{m-1})^n$ is written $\alpha \not\to (\beta_0, \cdots, \beta_{m-1})^n$.

Let α and β be ordinals. We say that α can be *pinned* to β, written $\alpha \twoheadrightarrow \beta$, iff there is $f\colon \alpha \to \beta$ such that for any set $X \subseteq \alpha$, if X has order type α then $\{f(\xi)\colon \xi \in X\}$ has order type β.

There is an important relation between pinning and partition relations, given by the following:

Proposition 1.1 *Suppose α and β are ordinals and $m < \omega$. If $\alpha \twoheadrightarrow \beta$ and $\beta \not\to (\beta, m)^2$, then $\alpha \not\to (\alpha, m)^2$.*

PROOF. Let $f\colon \alpha \to \beta$ be a pinning map. Suppose $[\beta]^2 = P_0 \cup P_1$ and P_0, P_1 are a counterexample to $\beta \to (\beta, m)^2$. Let $Q_1 = \{\{\xi, \eta\}\colon \xi, \eta < \alpha, f(\xi) \neq f(\eta)$ and $\{f(\xi), f(\eta)\} \in P_1\}$. Let $Q_0 = [\alpha]^2 - Q_1$. Then Q_0, Q_1 are a counterexample to $\alpha \to (\alpha, m)^2$.

REMARK. Obviously a more general theorem than Proposition 1.1 is true. However, in the rest of this paper we never need more than Proposition 1.1.

2. Statement of results

We shall be concerned with the classification of ordinals α which satisfy $\alpha \to (\alpha, 3)^2$. We treat particularly the case $\alpha = \kappa^2$, where κ is a cardinal.

It is easy to see that if $\alpha \to (\alpha, 3)^2$ then α must be indecomposable. If α is an infinite cardinal then it is well known that $\alpha \to (\alpha, \omega)^2$. See [5]. Specker ([15]) proved $\omega^2 \to (\omega^2, m)^2$ for all $m < \omega$, and $\omega^n \not\to (\omega^n, 3)^2$ for $3 \leqq n < \omega$. Chang ([1]) proved $\omega^\omega \to (\omega^\omega, 3)^2$ and Milner improved his result to $\omega^\omega \to (\omega^\omega, m)^2$ for all $m < \omega$. A much shorter proof of the latter result was found subsequently by Larson ([12]). Galvin and Larson (see [6]) have shown that if $\omega^\omega \leqq \alpha < \omega_1$ and $\alpha \to (\alpha, 3)^2$, then $\alpha = \omega^{\omega^\beta}$ for some β. It is still unsettled whether $\omega^{\omega^2} \to (\omega^{\omega^2}, 3)^2$.

For uncountable ordinals, less is known. Hajnal proved:

(1) if $2^\kappa = \kappa^+$ then $\kappa^+ \cdot \kappa \not\to (\kappa^+ \cdot \kappa, 3)^2$

(2) if $2^\kappa = \kappa^+$ and κ is regular, then $(\kappa^+)^2 \to ((\kappa^+)^2, 3)^2$.

The proof of (1) is in [2]; (2) is proved in [8]. In Section 3 we show how to extend Hajnal's method to prove (2) in case κ is singular. We obtain

THEOREM 1. *If* $2^\kappa = \kappa^+$ *then* $(\kappa^+)^2 \not\to ((\kappa^+)^2, 3)^2$.

Section 4 is devoted to a proof of the following:

THEOREM 2. *Assume that κ is a strong limit cardinal and $m < \omega$. Then* $\kappa^2 \to (\kappa^2, m)^2$ *iff* $(\mathrm{cf}\,\kappa)^2 \to ((\mathrm{cf}\,\kappa)^2, m)^2$.

If GCH is assumed, then Theorems 1 and 2 reduce the problem of determining when $\kappa^2 \to (\kappa^2, 3)^2$ to the case for inaccessible cardinals.

Further information is given by the following, which is proved in section 5.

THEOREM 3. *Assume κ is regular and $\kappa^2 \to (\kappa^2, 3)^2$. Then the κ-Souslin hypothesis holds.*

A cardinal κ is called *weakly compact* if $\kappa \to (\kappa, \kappa)^2$. It is known that if κ is weakly compact then $\kappa^2 \to (\kappa^2, m)^2$ for all $m < \omega$. See [7], for example.

In view of Jensen's result ([9]) that if $V = L$ then κ is weakly compact iff the κ-Souslin hypothesis holds, we have

COROLLARY 4. *If $V = L$ and κ is a cardinal, then $\kappa^2 \to (\kappa^2, 3)^2$ iff $\mathrm{cf}\kappa$ is weakly compact.*

Other results in the same direction are the following: If κ is weakly compact, $\alpha < \kappa$ and $\alpha \to (\alpha, m)^2$ then $\kappa \cdot \alpha \to (\kappa \cdot \alpha, m)^2$ (due to the author) and $\kappa^2 \cdot \alpha \to (\kappa^2 \cdot \alpha, m)^2$ (due to Larson). See [11]. Also, if κ is a Ramsey cardinal then $\kappa^\omega \to (\kappa^\omega, m)^2$ and $\kappa^\omega \cdot \omega \to (\kappa^\omega \cdot \omega, m)^2$ for all $m < \omega$. This is proved by Larson in [10].

Little else is known, even assuming GCH. For example, it is not known whether GCH (or even $V = L$) settles $\omega_2 \cdot \omega \to (\omega_2 \cdot \omega, 3)^2$. This is Problem 13 in [3].

If GCH is not assumed, then the only substitute we have at present for (1) and Theorem 1 is a result of Larson ([10]), which states that if κ is uncountable and regular then $\kappa^{\omega+1} \not\to (\kappa^{\omega+1}, 3)^2$.

In Section 6 we prove

THEOREM 5. *Assume κ is a cardinal such that $\mathrm{cf}\kappa > \omega$. Let α be an indecomposable ordinal such that $\kappa^\omega < \alpha < \kappa^+$ and $\mathrm{cf}\alpha = \lambda$. Then $\alpha \to \kappa^\omega \cdot \lambda$. If $\mathrm{cf}\kappa = \mu$, then $\alpha \to \mu^\omega \cdot \lambda$ also.*

By Proposition 1.1, Theorem 5 together with Larson's result (and the observation that $\alpha \not\to (\alpha, 3)^2$ if α is decomposable) yields

THEOREM 6. *Assume κ is a cardinal and $\kappa^\omega < \alpha < \kappa^+$. If $\mathrm{cf}\alpha = \mathrm{cf}\kappa > \omega$, then $\alpha \not\to (\alpha, 3)^2$.*

This leaves open many interesting questions. For instance, are the following propositions consistent with ZFC:

(a) $$\omega_1 \cdot \omega \to (\omega_1 \cdot \omega, 3)^2$$

(b) $$\omega_1^2 \to (\omega_1^2, 3)^2$$

(c) $$\omega_1^\omega \to (\omega_1^\omega, 3)^2 \ ?$$

In view of Theorems 1 and 3, it seems possible that Martin's Axiom ([13]) may provide an approach to (b).

Sections 3–6 may be read independently of one another.

3. Proof of Theorem 1

In view of Hajnal's result, we may assume κ is singular.

Let $A = \{(\eta, \xi): \eta < \xi < \kappa^+\}$. Let $<_1$ be the lexicographical ordering of A and let $<_2$ be the backwards lexicographical ordering of A (i.e. the ordering by *second* element first). Then the order type of $(A, <_1)$ is $(\kappa^+)^2$ and the type of $(A, <_2)$ is κ^+.

If $X \subseteq A$ and $\alpha < \kappa^+$, then α is a *double limit point* of X iff $\{\eta < \alpha: \{\xi < \alpha: (\eta, \xi) \in X\}$ *is cofinal in* $\alpha\}$ is cofinal in α. Let $D = \{X \subseteq A: |X| = \kappa$ and $\{\mathrm{cf}\alpha$: α *is a double limit point of* $X\}$ *is cofinal in* $\kappa\}$. Since $|D| = 2^\kappa$ and $2^\kappa = \kappa^+$, we may assume $D = \{D_\alpha: \alpha < \kappa^+\}$.

Now, by induction on $<_2$, for each $(\eta, \xi) \in A$ we define a set $A(\eta, \xi) \subseteq A$ with the following properties:

(1) if $(\eta', \xi') \in A(\eta, \xi)$, then $\eta < \eta' < \xi' < \xi$

(2) if $x, y \in A(\eta, \xi)$, then $x \notin A(y)$ and $y \notin A(x)$

(3) $A(\eta, \xi)$ is a function

(4) if $\alpha < \xi$ and $D_\alpha \subseteq \{(\eta', \xi'): \eta < \eta' < \xi' < \xi\}$, then $D_\alpha \cap A(\eta, \xi) \neq 0$.

Let $x = (\eta, \xi)$ and suppose $A(y)$ has been defined for all $y <_2 x$. Let $E = \{D_\alpha: \alpha < \xi$ and $D_\alpha \subseteq \{(\eta', \xi'): \eta < \eta' < \xi' < \xi\}\}$, and suppose $|E| = \lambda$. Let $\langle E_\beta: \beta < \lambda\rangle$ be an enumeration of E. We will put $A(x) = \{x_\beta: \beta < \lambda\}$, where the $x_\beta \in E_\beta$ are defined by induction as follows. Let $x_\gamma = (\eta_\gamma, \xi_\gamma)$ for all $\gamma < \beta$. Let α be a double limit point of E_β such that $\mathrm{cf}\alpha > |\beta|$. This is possible since $|\beta| < \lambda \leqq \kappa$. Since $\mathrm{cf}\alpha > |\beta|$, there exists $\eta_\beta < \alpha$ satisfying

(5) $\{\xi' < \alpha: (\eta_\beta, \xi') \in E_\beta\}$ is cofinal in α

(6) for all $\gamma < \beta$, either $\alpha < \eta_\gamma$ or $\xi_\gamma < \eta_\beta$ or $\eta_\gamma < \eta_\beta < \alpha < \xi_\gamma$.

Also, since $\mathrm{cf}\alpha > |\beta|$, there exists $\xi_\beta < \alpha$ such that

(7) $(\eta_\beta, \xi_\beta) \in E_\beta$

(8) for all $\gamma < \beta$, if $\eta_\gamma < \eta_\beta < \alpha < \xi_\gamma$ then $(\eta_\beta, \xi_\beta) \notin A(\eta_\gamma, \xi_\gamma)$.

Of course, in order to make (8) true, we must use the inductive hypothesis (3) for each $(\eta_\gamma, \xi_\gamma)$, $\gamma < \beta$. Finally, let $x_\beta = (\eta_\beta, \xi_\beta)$. It is easy to check that (1)–(4) hold for $A(x_\beta)$.

Let $P_0 = \{\{x, y\} \in [A]^2: x \in A(y)\}$. It is clear from (1) and (2) that if B is a three-element subset of A then $[B]^2 \not\subseteq P_0$. Now suppose $X \subseteq A$ has order type $(\kappa^+)^2$ with respect to $<_1$. We will show that $[X]^2 \cap P_0 \neq 0$.

LEMMA 3.1. *If $X \subseteq A$ has order type $(\kappa^+)^2$, then the set of double limit points of X contains a set which is closed and unbounded in κ^+.*

PROOF. Let $X' = \{\eta : \{\xi : (\eta, \xi) \in X\}$ *is cofinal in* $\kappa^+\}$. Then X' is cofinal in κ^+, so Y', the set of limit points of X', is closed and unbounded in κ^+. Similarly, if for each $\eta \in X'$ we let $X_\eta = \{\xi : (\eta, \xi) \in X\}$, then Y_η, the set of limit points of X_η, is closed and unbounded in κ^+. Then $Z = Y' \cap \{\alpha :$ *for all* $\eta < \alpha,\ \alpha \in Y_\eta\}$ is closed and unbounded. Moreover every element of Z is a double limit point of X, so the lemma is proved.

Now choose η such that $|\{\xi : (\eta, \xi) \in X\}| = \kappa^+$. By the lemma there is a closed unbounded set C consisting of double limit points of $\{(\eta', \xi') : \eta < \eta'$ *and* $(\eta', \xi') \in X\}$. Hence there exists some $\beta < \kappa^+$ such that $\{\mathrm{cf}\,\alpha : \alpha \in C \cap \beta\}$ is cofinal in κ. Let $Y = \{(\eta', \xi') \in X : \eta < \eta'$ *and* $\xi' < \beta\}$. Then $Y \in D$. Say $Y = D_\alpha$. Let $\xi > \alpha$ be such that $(\eta, \xi) \in X$. Then by (4) we have $A(\eta, \xi) \cap D_\alpha \neq 0$. It follows immediately that $[X]^2 \cap P_0 \neq 0$, and the proof is complete.

4. Proof of Theorem 2

The theorem is a tautology if κ is regular, so we assume κ is singular. We give the proof for $m = 3$; the rest is left to the reader.

Let $\lambda = \mathrm{cf}\,\kappa$, and let $\langle \kappa_\alpha : \alpha < \lambda \rangle$ be an increasing sequence of regular cardinals such that $\Sigma_{\alpha < \lambda} \kappa_\alpha = \kappa$.

Let $A = \{(\alpha, \beta) : \alpha, \beta < \kappa\}$ and let $B = \{(\alpha, \beta) : \alpha, \beta < \lambda\}$. Under the lexicographical ordering, A has order type κ^2 and B has type λ^2.

First we prove that $\kappa^2 \to (\kappa^2, 3)^2$ implies $\lambda^2 \to (\lambda^2, 3)^2$. Let $\{S_\alpha : \alpha < \lambda\}$ be a collection of disjoint sets such that

(1) $|S_\alpha| = \kappa_\alpha$

(2) $\cup \{S_\alpha : \alpha < \lambda\} = \kappa$

(3) if $\alpha < \beta$, $\xi \in S_\alpha$ and $\eta \in S_\beta$, then $\xi < n$.

Now suppose $f : [B]^2 \to 2$ is a counterexample to $\lambda^2 \to (\lambda^2, 3)^2$. Define $g : [A]^2 \to 2$ as follows: If $\xi \in S_\alpha$, $\eta \in S_\beta$, $\xi' \in S_{\alpha'}$, $\eta' \in S_{\beta'}$ and $(\alpha, \beta) \neq (\alpha', \beta')$, then let $g(\{(\xi, \eta), (\xi', \eta')\}) = f(\{(\alpha, \beta), (\alpha', \beta')\})$; otherwise let $g(\{(\xi, \eta), (\xi', \eta')\}) = 1$. It is easy to see that g is a counterexample to $\kappa^2 \to (\kappa^2, 3)^2$.

Now we prove that $\lambda^2 \to (\lambda^2, 3)^2$ implies $\kappa^2 \to (\kappa^2, 3)^2$. Let $f : [A]^2 \to 2$, and assume that there is no three-element set $X \subseteq A$ such that $f(x) = 1$ for all $x \in [X]^2$. Let $g : [\kappa]^4 \to 8$ be such that if $\alpha < \beta < \gamma < \delta < \kappa$ and $\alpha' < \beta' < \gamma' < \delta' < \kappa$, then

iff
$$g(\{\alpha,\beta,\gamma,\delta\}) = g(\{\alpha',\beta',\gamma',\delta'\})$$

$$f(\{(\alpha,\beta),(\gamma,\delta)\}) = f(\{(\alpha',\beta'),(\gamma',\delta')\}),$$

$$f(\{(\alpha,\gamma),(\beta,\delta)\}) = f(\{(\alpha',\gamma'),(\beta',\delta')\}) \quad \text{and}$$

$$f(\{(\alpha,\delta),(\beta,\gamma)\}) = f(\{(\alpha',\delta'),(\beta',\gamma')\}).$$

LEMMA 4.1. *There is a collection* $\{S_\alpha : \alpha < \lambda\}$ *which satisfies* (1) *and* (3), *and in addition for any* $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \lambda$, *if* $x_i, x'_i \subseteq S_{\alpha_i}$ *and* $|x_i| = |x'_i|$ *for* $i = 1,2,3,4$, *and if* $|\bigcup\{x_i : 1 \leqq i \leqq 4\}| = 4$, *then*

$$g(x_1 \cup x_2 \cup x_3 \cup x_4) = g(x'_1 \cup x'_2 \cup x'_3 \cup x'_4).$$

PROOF. This is an immediate consequence of the Canonization Lemma of Erdös, Hajnal and Rado (Lemma 3 on p. 110 of [4]). The proof in [4] uses the GCH, but for strong limit cardinals that assumption is unnecessary.

Let Z_α, $\alpha < \lambda$, be such that

(4) for all $\xi \in Z_\alpha$, $\alpha < \xi < \lambda$

(5) if $\alpha \neq \beta$ then $Z_\alpha \cap Z_\beta = 0$.

(6) $|Z_\alpha| = \lambda$.

Let $W = \{(\alpha,\beta),\ \alpha \in Z_0 \text{ and } \beta \in Z_\alpha\}$. Then W, with the lexicographical ordering, has type λ^2. Furthermore W has the convenient property that if (α,β) and (α',β') are distinct members of W, then either $\alpha = \alpha'$ and α,β and β' are distinct or else α, α', β, β' are all distinct.

Now define $h: [W]^2 \to 2$ as follows. Let $(\alpha,\beta) < (\alpha',\beta')$ lexicographically. Then $h(\{(\alpha,\beta),(\alpha',\beta')\}) = f(\{(\xi,\eta),(\xi',\eta')\})$, where $\xi < \xi'$, $\xi \in S_\alpha$, $\xi' \in S_{\alpha'}$, $\eta \in S_\beta$ and $\eta' \in S_{\beta'}$. By Lemma 4.1, the definition of h is independent of the choice of ξ, ξ', η and η'.

Suppose X is a three-element subset of W and $h(x) = 1$ for all $x \in [W]^2$. Say $X = \{(\alpha_i,\beta_i): i < 3\}$ and $(\alpha_0,\beta_0) < (\alpha_1,\beta_1) < (\alpha_2,\beta_2)$. Then choose ξ_i, η_i, $i < 3$, so that $\xi_0 < \xi_1 < \xi_2$ and $\xi_i \in S_{\alpha_i}$, $\eta_i \in S_{\beta_i}$, $i < 3$. Then $f(x) = 1$ for all $x \in [\{(\xi_i,\eta_i): i < 3\}]^2$, contradicting our assumption concerning f. Since $\lambda^2 \to (\lambda^2, 3)^2$, we conclude that there exists a set $X \subseteq W$ of order type λ^2 such that $h(x) = 0$ for all $x \in [X]^2$.

Let $Y = \{\alpha : |\{\beta : (\alpha,\beta) \in X\}| = \lambda\}$. For $\alpha \in Y$, let $X_\alpha = \{\beta : (\alpha,\beta) \in X\}$. Fix $\alpha \in Y$ and define $k_\alpha : [X_\alpha]^2 \to 2$ by $k_\alpha(\{\beta,\gamma\}) = (f(\{(\xi_1,\eta_1),(\xi_2,\eta_2)\}), f(\{(\xi'_1,\eta'_1),(\xi'_2,\eta'_2)\}))$, where $\beta < \gamma$, $\xi_1,\xi_2,\xi'_1,\xi'_2 \in S_\alpha$, $\eta_1,\eta'_2 \in S_\beta$, $\eta_2,\eta'_1 \in S_\gamma$, $\xi_1 < \xi_2$ and $\xi'_1 < \xi'_2$.

LEMMA 4.2. *There is $X'_\alpha \subseteq X_\alpha$ such that $|X'_\alpha| = \lambda$ and $k_\alpha(x) = (0,0)$ for all $x \in [X'_\alpha]^2$.*

PROOF. Since $\lambda \to (\lambda, \omega, \omega, \omega)^2$ (see [5], theor. 44), we know that if Lemma 4.2 is false then there exist m, n and $X''_\alpha \subseteq X_\alpha$ such that X''_α is infinite, $k_\alpha(x) = (m, n)$ for all $x \in [X''_\alpha]^2$, and either $m = 1$ or $n = 1$. Suppose $m = 1$; the proof for $n = 1$ is similar. Let $\beta_0 < \beta_1 < \beta_2$ be the first three members of X''_α. Let $\xi_0 < \xi_1 < \xi_2$ be members of S_α and let $\eta_i \in S_{\beta_i}$, $i < 3$. Then clearly $f(\{(\xi_i, \eta_i), (\xi_j, \eta_j)\}) = 1$ whenever $i < j < 3$, contradicting our assumption concerning f. Hence Lemma 4.2 is true.

For $\alpha \in Y$ and $\xi \in S_\alpha$, define $U_\xi \subseteq \cup\{S_\beta \colon \beta \in X'_\alpha\}$ so that

(7) $U_\xi \cap U_{\xi'} = 0$ if $\xi, \xi' \in S_\alpha$ and $\xi \neq \xi'$

(8) $|U_\xi| = \kappa$.

Let $T = \cup_{\alpha \in Y} \cup_{\xi \in S_\alpha} \{(\xi, \eta) \colon \eta \in U_\xi\}$. Since $\kappa \to (\kappa, 3)^2$, we may assume that $f(x) = 0$ whenever ξ is fixed and $x \in [\{(\xi, \eta) \colon \eta \in U_\xi\}]^2$. We assert that $f(x) = 0$ for all $x \in [T]^2$. Since T has type κ^2, this will complete the proof.

Let $x = \{(\xi, \eta), (\xi', \eta')\} \in [T]^2$. Let $\xi \in S_\alpha$, $\xi' \in S_{\alpha'}$, $\eta \in S_\beta$, $\eta' \in S_{\beta'}$. We may assume $\xi < \xi'$.

Case 1. $\alpha \neq \alpha'$. Then α, α', β, β' are distinct and $f(x) = h(\{(\alpha, \beta), (\alpha', \beta')\}) = 0$ since X is homogeneous for h.

Case 2. $\alpha = \alpha'$ and $\beta \neq \beta'$. Then we are done by Lemma 4.2 and the definition of X'_α.

Case 3. $\alpha = \alpha'$ and $\beta = \beta'$. Assume $f(x) = 1$. Let $\xi_0 < \xi_1 < \xi_2$ be members of S_α and let $\eta_0 < \eta_1 < \eta_2$ be members of S_β. If $\eta < \eta'$ then, by Lemma 4.1 and the definition of g, $f(y) = 1$ for all $y \in [\{(\xi_0, \eta_0), (\xi_1, \eta_1), (\xi_2, \eta_2)\}]^2$, while if $\eta' < \eta$ then $f(y) = 1$ for all $y \in [\{(\xi_0, \eta_2), (\xi_1, \eta_1), (\xi_2, \eta_0)\}]^2$. In either case our assumption concerning f is contradicted. Hence $f(x) = 0$.

5. Proof of Theorem 3

We will show that if κ is regular and a κ-Souslin tree exists, then $\kappa^2 \to (\kappa^2, 3)^2$.

Suppose $(T, <_T)$ is a κ-Souslin tree. Since $|T| = \kappa$, we may assume $T = \kappa$. Let $X = \{(\alpha, \beta) \colon \alpha <_T \beta\}$. Let X be ordered lexicographically with respect to the usual ordering on κ, i.e., let $(\alpha, \beta) < (\gamma, \delta)$ iff $\alpha < \gamma$ or $\alpha = \gamma$ and $\beta < \delta$. Then X has order-type κ^2.

Next we define sets P_0, P_1 so that $[X]^2 = P_0 \cup P_1$. Let P_1 be the set of all pairs $\{(\alpha,\beta),(\alpha',\beta')\} \in [X]^2$ such that $\alpha <_T \alpha' <_T \beta$ and for all γ, if $\alpha' <_T \gamma \leqq_T \beta$ then $\gamma \not\leqq_T \beta'$. Let $P_0 = [X]^2 - P_1$.

It is clear that there is no three-element set $Z \subseteq X$ with $[Z]^2 \subseteq P_1$. It remains to show that if $Y \subseteq X$ and Y has order-type κ^2, then $[Y]^2 \cap P_1 \neq 0$.

Let $A_0 = \{\alpha : |\{\beta : (\alpha,\beta) \in Y\}| = \kappa\}$. Then $|A_0| = \kappa$. Let $T_0 = \{\alpha : \exists \beta \in A_0\, \alpha \leqq_T \beta\}$ and let $B_0 = \{\alpha : \alpha \notin T_0$ *but for all* $\beta <_T \alpha, \beta \in T_0\}$. Then B_0 is an antichain in $(T, <_T)$, so $|B_0| < \kappa$. Choose $\alpha_0 \in A_0$ so that $l(\alpha_0) > l(\beta)$ for all $\beta \in B_0$. Then it must be true that for all $\beta >_T \alpha_0$ there is $\gamma \in A_0$ such that $\beta \leqq_T \gamma$.

Now let $A_1 = \{\beta : (\alpha_0, \beta) \in Y\}$, let $T_1 = \{\alpha : \exists \beta \in A_1\, \alpha \leqq_T \beta\}$ and let $B_1 = \{\alpha : \alpha \notin T_1$ *but for all* $\beta <_T \alpha, \beta \in T_1\}$. Then B_1 is an antichain so $|B_1| < \kappa$. Choose $\alpha'_0 \in T_1$ so that $l(\alpha'_0) > l(\beta)$ for every $\beta \in B_1$. As before, for all $\beta >_T \alpha'_0$ there is $\gamma \in A_1$ such that $\beta \leqq_T \gamma$.

Now let $\alpha_1 \in A_0$ be such that $\alpha'_0 \leqq_T \alpha_1$, and let β_1 be such that $(\alpha_1, \beta_1) \in Y$. Since α_1 has at least two immediate successors in T, there is an immediate successor α'_1 of α_1 which is incomparable with β_1. Let $\beta_0 \in A_1$ be such that $\alpha'_1 \leqq_T \beta_0$. Then $\{(\alpha_0,\beta_0),(\alpha_1,\beta_1)\} \in P_1$, as desired. Hence $\kappa^2 \not\to (\kappa^2, 3)^2$.

6. Proof of Theorem 5

Throughout this section, if κ is a cardinal and $n < \omega$ then let ${}^n\kappa = \{(\alpha_1, \cdots, \alpha_n) : \alpha_1, \cdots, \alpha_n < \kappa\}$. When ${}^n\kappa$ is ordered lexicographically it has order type κ^n. Note that ${}^1\kappa = \{(\alpha) : \alpha < \kappa\}$.

LEMMA 6.1. *Let κ be an infinite cardinal and let $1 \leqq n < \omega$. Then no set of order type κ^n may be decomposed into fewer than* $\mathrm{cf}\,\kappa$ *sets of smaller type.*

PROOF. It will suffice to show that if $\lambda < \mathrm{cf}\,\kappa$, $X_\alpha \subseteq {}^n\kappa$ for each $\alpha < \lambda$ and each X_α has type $< \kappa^n$, then $\cup\{X_\alpha : \alpha < \lambda\} \neq {}^n\kappa$.

By induction on n we define the notion of a *large* subset of ${}^n\kappa$. We say $X \subseteq {}^1\kappa$ is large iff $|\kappa - \{\alpha : (\alpha) \in X\}| < \kappa$. If $X \subseteq {}^{n+1}\kappa$ then X is large iff $|\kappa - \{\alpha_1 : \{(\alpha_2, \cdots, \alpha_{n+1}) : (\alpha_1, \alpha_2, \cdots, \alpha_{n+1}) \in X\}$ is a large subset of ${}^n\kappa\}| < \kappa$. It is easy to check the following by induction on n:

(a) If $\mu < \mathrm{cf}\,\kappa$ then the intersection of μ large subsets of ${}^n\kappa$ is large

(b) If $X \subseteq {}^n\kappa$ and X has order type $< \kappa^n$ then ${}^n\kappa - X$ is large.

Lemma 6.1 is an immediate consequence of (a) and (b).

LEMMA 6.2. *Suppose κ is a singular cardinal and* $\mathrm{cf}\,\kappa = \lambda > \omega$. *Then* $\kappa^\omega \to \lambda^\omega$. *In fact, there is a function* $f: \kappa^\omega \to \lambda^\omega$ *such that for all* $n < \omega$, *if* $X \subseteq \kappa^\omega$ *is of order type* κ^n *then* $\{f(\alpha): \alpha \in X\}$ *has order type* $\geqq \lambda^n$.

PROOF. It will suffice to show that for each $n < \omega$ there is a function $f_n: \kappa^n \to \lambda^n$ such that for all $m \leqq n$ and all $X \subseteq \kappa^n$, if X has order type κ^m then $\{f_n(\alpha): \alpha \in X\}$ has order type $\geqq \lambda^m$. The function f is then obtained by patching together copies of the functions f_n in the following manner: κ^ω may be written as the disjoint union of sets C_n, $n < \omega$, such that each C_n has order type κ^n. Let $f'_n: C_n \to \lambda^n$ be a copy of f_n and let $f = \bigcup_{n<\omega} f'_n$. If $C \subseteq \kappa^\omega$ has order type κ^n then by Lemma 6.1 there is some p so that $C \cap C_p$ has order type κ^n, and since $\{f'_p(\alpha): \alpha \in C \cap C_p\}$ has order type $\geqq \kappa^n$ we are done.

Let $\langle \kappa_\alpha: \alpha < \lambda \rangle$ be an increasing sequence of cardinals such that $\sup\{\kappa_\alpha: \alpha < \lambda\} = \kappa$. Define $h: \kappa \to \lambda$ by $h(\alpha) = \beta$ iff β is the least ordinal such that $\alpha < \kappa_\beta$. Define $g_n: {}^n\kappa \to {}^n\lambda$ by $g_n(\alpha_1, \cdots, \alpha_n) = (h(\alpha_1), \cdots, h(\alpha_n))$. We will prove by induction on n that for all $m \leqq n$ and all $X \subseteq {}^n\kappa$, if X has type κ^m then $\{g_n(x): x \in X\}$ has type $\geqq \lambda^m$, and this will prove the lemma.

For $n = 1$ this is clear. Assume $n > 1$.

If $m = 1$ then again the assertion is clear. Suppose $m > 1$ and for all smaller values of m the assertion is true. Let $X \subseteq {}^n\kappa$ have order type κ^m.

Case 1. X is bounded in ${}^n\kappa$. Then it is not difficult to see that there are sequences $\langle \xi_\alpha: \alpha < \lambda \rangle$ and $\langle A_\alpha: \alpha < \lambda \rangle$ such that

(1) $A_\alpha \subseteq {}^{n-1}\kappa$ for all α

(2) A_α has order type $\geqq \kappa^{m-1} \cdot \kappa_\alpha$

(3) $\{(\xi_\alpha, \beta_2, \cdots, \beta_n): (\beta_2, \cdots, \beta_n) \in A_\alpha\} \subseteq X$ for all α.

(Note: we do *not* require the ξ_α and A_α to be distinct.) Moreover, we may assume $h(\xi_\alpha) = h(\xi_\beta)$ for all $\alpha < \beta < \lambda$. It follows that the order type of $\{g_n(x): x \in X\}$ is at least as large as the order type of $\{g_{n-1}(y): y \in \bigcup\{A_\alpha: \alpha < \lambda\}\}$. However, since each A_α has type $\geqq \kappa^{m-1} \cdot \kappa_\alpha$, it follows that $\bigcup\{A_\alpha: \alpha < \lambda\}$ has type $\geqq \kappa^m$ and hence by inductive hypothesis the order type of $\{g^{n-1}(y): y \in \bigcup\{A_\alpha: \alpha < \lambda\}\}$ is at least λ^m.

Case 2. X is cofinal in ${}^n\kappa$. For each $\alpha < \lambda$, let $B_\alpha = X \cap \{(\beta_1, \beta_2, \cdots, \beta_n): \kappa_\alpha \leqq \beta_1 < \kappa_{\alpha+1}, \beta_2, \cdots, \beta_n < \kappa\}$. Then $\{\alpha: B_\alpha$ *has type* $\geqq \kappa^{m-1}\}$ has cardinality λ. By our assumption on m, $\{g_n(x): x \in B_\alpha\}$ has type $\geqq \lambda^{m-1}$ for each B_α of type $\geqq \kappa^{m-1}$. But then $\{g_n(x): x \in X\}$ has type $\geqq \lambda^{m-1} \cdot \lambda = \lambda^m$.

For the next lemma, we need a result ([14]) which has been called the Milner-Rado "paradox", namely:

Let α be an ordinal number and let κ be a cardinal such that $\kappa \leqq \alpha < \kappa^+$. Then there are sets A_n for each $n < \omega$ such that $\alpha = \bigcup\{A_n : n < \omega\}$ and each A_n has order type less than κ^ω.

LEMMA 6.3. *Let κ be a cardinal such that* $\mathrm{cf}\kappa > \omega$, *and let $\kappa^\omega \leqq \alpha < \kappa^+$. Then there is $f_\alpha : \alpha \to \kappa^\omega$ such that for every $A \subseteq \alpha$ and every $n < \omega$, if A has order type κ^n then $\{f_\alpha(\beta) : \beta \in A\}$ has order type $\geqq \kappa^n$. Therefore if $A \subseteq \alpha$ has order type κ^ω then so does $\{f_\alpha(\beta) : \beta \in A\}$. In particular, $\alpha \to \kappa^\omega$.*

PROOF. By the Milner-Rado "paradox", we may write $\alpha = \bigcup\{A_n : n < \omega\}$ where the A_n are disjoint and of order type less than κ^ω. Therefore there is a function $f_\alpha : \alpha \to \kappa^\omega$ such that

(4) for each n, $f_\alpha \mid A_n$ is one-to-one and order-preserving, and

(5) if $m < n$, $\gamma \in A_m$ and $\delta \in A_n$, then $f_\alpha(\gamma) < f_\alpha(\delta)$.

Now suppose $A \subseteq \alpha$ has order type κ^n. If $\{f_\alpha(\beta) : \beta \in A\}$ has order type $< \kappa^n$, then clearly $A \cap A_m$ has order type $< \kappa^n$ for each m, and hence by Lemma 6.1 A has type $< \kappa^n$, a contradiction.

Now we prove Theorem 5. Assume κ is a cardinal, $\mathrm{cf}\kappa > \omega$, $\kappa^\omega < \alpha < \kappa^+$, α is indecomposable and $\mathrm{cf}\alpha = \lambda$. We want to show $\alpha \to \kappa^\omega \cdot \lambda$. If $\alpha = \kappa^\omega \cdot \lambda$ this is trivial; assume $\alpha > \kappa^\omega \cdot \lambda$.

Since α is indecomposable there is an increasing sequence $\langle \xi_\beta : \beta < \lambda \rangle$ such that $\xi_0 = 0$, $\xi_{\beta+1} \geqq \xi_\beta + \kappa^\omega$ for all β, $\xi_\beta = \sup\{\xi_\gamma : \gamma < \beta\}$ whenever β is a limit ordinal, and $\sup\{\xi_\beta : \beta < \lambda\} = \alpha$. Let $Z_\beta = \{\xi : \xi_\beta \leqq \xi < \xi_{\beta+1}\}$ for each $\beta < \lambda$. By Lemma 6.3 there is a function $g_\beta : Z_\beta \to \kappa^\omega$ such that every subset of Z_β of order type κ^ω has an image of order type $\geqq \kappa^\omega$.

Notice that the set $\lambda \times \kappa^\omega$, ordered lexicographically, has order type $\kappa^\omega \cdot \lambda$. Define $g : \alpha \to \lambda \times \kappa^\omega$ by letting $g(\xi) = (\beta, g_\beta(\xi))$ if $\xi \in Z_\beta$. Now let $Z \subseteq \alpha$ have order type α. We claim $\{g(\xi) : \xi \in Z\}$ has order type $\kappa^\omega \cdot \lambda$. Let $B = \{\beta : Z \cap Z_\beta$ *has order type* $\geqq \kappa^\omega\}$. If $|B| = \lambda$ we will clearly be done. If $|B| < \lambda$ then there is some $\gamma < \lambda$ so that $B \subseteq \gamma$. But then Z must have order type $\leqq \xi_\gamma + \kappa^\omega \cdot \lambda$, and since α is indecomposable this means $\alpha = \kappa^\omega \cdot \lambda$, contrary to our assumption.

All that remains is the second assertion of Theorem 5. Assume now that $\mathrm{cf}\kappa = \mu < \kappa$. It will suffice to show $\kappa^\omega \cdot \lambda \to \mu^\omega \cdot \lambda$. If $\mu < \lambda$ then $\mu^\omega \cdot \lambda = \lambda$

and the assertion is obvious. Assume $\mu \geqq \lambda$. Let $f\colon \kappa^{\omega} \to \mu^{\omega}$ be as in Lemma 6.2. Define $g\colon \lambda \times \kappa^{\omega} \to \lambda \times \mu^{\omega}$ by $g(\beta, \gamma) = (\beta, f(\gamma))$. It is straightforward to check that g works.

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